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Comment on: Ising Models on Hyperbolic Graphs

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New proofs are given for Propositions 1 and 2 of C. C. Wu, *J. Stat. Phys.* **50**:251 (1996). The propositions involved upper and lower bounds on the critical temperature for these models. Besides being more direct than the previous proofs, the new proofs improve both bounds.

KEY WORDS: Ising model; hyperbolic graphs; correlation inequalities.

In a very recent article in this journal Wu[1] considered ferromagnetic Ising models on hyperbolic graphs. He presented four propositions regarding these systems which were basically proven using results from percolation theory. In particular in his first proposition he established an upper bound on the critical temperature for these Ising systems while in the second proposition he followed this up by proving a lower bound on the critical temperature. We present alternate proofs of his first two propositions. The proofs may be considered more direct in that they do not resort to results from percolation theory. Also the proofs improve both of his bounds.

The system consists of a ferromagnetic Ising model with nearest neighbor interactions on a graph **G**. These graphs were defined in the following manner. "Let $\mathbf{T}'_k$ to be a homogeneous tree with degree k (i.e., each site of $\mathbf{T}'_k$ has exactly $k+1$ neighbors) and let 0 be the origin of $\mathbf{T}'_k$. Define $\mathbf{T}_k$ to be the "forwarding" tree obtained by deleting one of the $k+1$ edges emanating from 0. Then **G** is the graph obtained by adding to $\mathbf{T}_k$

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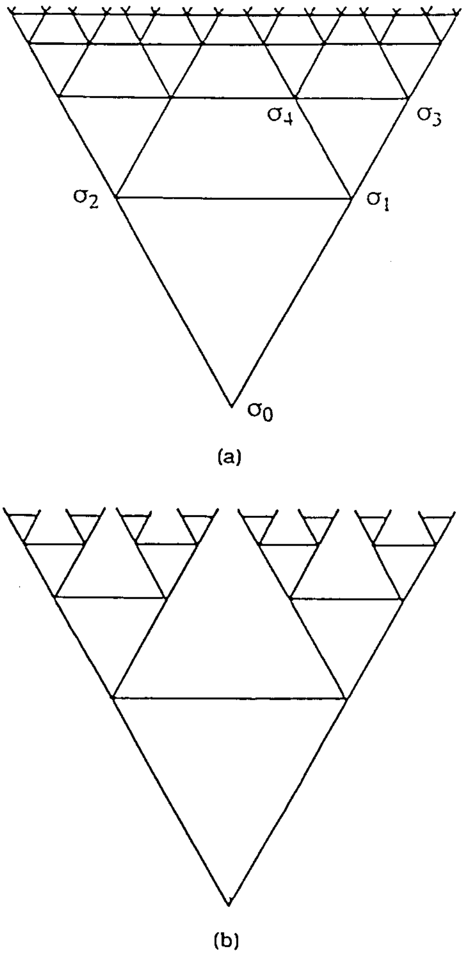

Fig. 1. (a) The first levels of the hyperbolic graph for $k=2$. (b) The first four levels of the Husimi using the three site triangle as the basic building block.

edges connecting equal-level sites of $\mathbf{T}_k$."[1] See Fig. 1a for the $k=2$ case. We take the Hamiltonian to be

$$\mathscr{H} = -J \sum_{\{i,\,k\}} \sigma_i \sigma_k \tag{1}$$

where the sum is over the bonds of $\mathbf{G}$. We take $J \geqslant 0$ and $\sigma = \pm 1$. Note, we do not have a factor of 1/2 which is in ref. 1 as we feel the above is simpler when dealing with the spin system. We compute thermal averages in the standard manner, denote them by brackets, $\langle\rangle$, and take $\beta = 1/kT$. We define $M = \langle \sigma_0 \rangle_+$ with the subscript 0 denoting the base or "root" site of the tree, and with the subscript + denoting plus boundary conditions. We begin with Wu's Proposition 2.

Proposition 2. If $\beta J > (1/2) \ln[(k+1)/(k-1)]$, then $M > 0$ and consequently there is more than one Gibbs state.

Proof of Proposition 2. One sees from Fig. 1a that if the horizontal interactions are deleted one has a Bethe lattice. The thermal average of the base or "root" site, σ_0, can be computed exactly. One very easy method to compute $\langle \sigma_0 \rangle$ is to treat the system as a dynamical system. See Eggarter[2] or Thompson[3] for this approach. One has for $\beta J > (1/2) \ln[(k+1)/(k-1)]$ that $M > 0$. The fact that the base site has k nearest neighbors or $k+1$ is irrelevant. From the classic correlation inequalities of Griffiths, Kelly and Sherman,[5, 5, 6] in particular the second Griffiths, Kelly, Sherman inequality (hereafter GKS II) one knows that adding ferromagnetic interactions to get back to the full graph **G** can not decrease the value of M hence the proposition is proved. ∎

Now one can do considerably better than the above lower bound for the critical temperature by deleting only a subset of all the horizontal interactions shown in Fig. 1. This has an importance beyond simply having a better bound in that if one has a lower bound less than $(1/2) \ln(k)/k$ then there is a second phase transition characterized by behavior of the two-point function. For details see ref. 1. Wu's bounds as stated above produce such a situation for $k \geqslant 8$. By deleting only some horizontal bonds we bring this down to $k \geqslant 5$. Wu states in ref. 1 that the occurrence of this second phase transition it is believed to occur for $k \geqslant 2$.

In particular, if one deletes only those interactions resulting in the graph **G**′ shown in Fig. 1b for $k = 2$ and does analogous deletions for other k values then one has a Husimi tree rather than a Bethe lattice. The basic buiding block for $k = 2$ is a three site triangle consisting of a base site interacting with two sites on the next higher level and their horizontal interaction. Generally it consists of a single base site interacting with k other sites on the next higher level along with their horizontal interactions. Using the same approach as with the Bethe lattice it is relatively simple to calculate the temperature below which $M > 0$. The calculations involve a one dimensional, dynamical system consisting of a rational function as does the calculations involving the Bethe lattice. The larger the value of k and the fewer the number of horizontal interactions deleted the more complex this rational function becomes. The values of our bounds using this approach for $k = 2, 4, 5, 6$, and $k = 8$ are given in Table 1. We denote these approximations as First Level Husimi Tree approximations. The one dimensional map governing each of these systems was found using Mathematica as was the temperature at which $M > 0$. This for larger k systems involves finding the zeros of a polynomial in $\exp[2\beta J]$ and since the degree of the polynomials is greater than five must be done numerically. Even for the $k = 2$ case it involves a fourth degree polynomial and the analytic expression of the solution is not very illuminating.

Finally, in regards to lower bounds on T_c, still further improvement can be had by considering still less deletion of horizontal interactions or equivalently using larger building blocks for the generation of the Husimi trees. At the bottom of Table 1 results for the $k = 2$ case are given using building blocks with three levels, four levels and five levels of sites in the basic building block. Despite this sequence of improving bounds we have not been able to establish the existence of Wu's second phase transition for $k < 5$.

We now present an improvement of Wu's Proposition 1 along with our proof. This is in essence an upper bound on the critical temperature and our methods of proof are based on earlier works of the author.[7]

Proposition 1. If βJ is such that $\tanh(\beta J) < 1/(k+3)$ then $M = 0$ and consequently the Gibbs state is unique.

Proof of Proposition 1. First we remark that it will make our work somewhat simpler if we add on each horizontal row, except at the root site level, an interaction between the right most site on a row with the left most site on the same row. This means that a spin on any site except the root site has $k + 3$ interactions. If we can find a temperature above which $M = 0$ for this system then by GKS II $M = 0$ for the original system.

Table 1. Lower and Upper Bounds on T_c

	Lower bounds on T_c		Upper bounds on T_c	
k	Reference (1)	This Paper[a]	Reference (1)	This paper
2	1.8204	2.4853[b]	6.9522	4.9327
4	3.9152	5.1699	10.9697	6.9522
5	4.9326	6.2133	12.9744	7.9582
6	5.9440	7.4167	14.9778	8.9628
8	7.6944	9.5462	18.9825	9.5462

[a] Using the first level Husimi tree approximations.

[b] Using second, third, and fourth level Husimi tree approximations one has 2.7332, 2.9106, and 3.0333.

For Ising spins one has the identity

$$\exp[\beta J\sigma_k\sigma_l] = \cosh(\beta J)[1 + \tanh(\beta J)\,\sigma_k\sigma_l] \tag{2}$$

where hereafter we set $T = \tanh(\beta J)$. Applying this identity to the thermal average of some σ_i one obtains

$$\langle\sigma_i\rangle = \frac{\langle\sigma_j\rangle_{jk} + T\langle\sigma_i\sigma_j\sigma_k\rangle_{jk}}{1 + T\langle\sigma_j\sigma_k\rangle} \tag{3}$$

where the subscript jk on the brackets denotes a thermal average where the interaction between the jth and kth sites has been deleted. One may use the identity (2) again to delete other interactions,

For simplicity we will suppose that $k = 2$. Then for the root site in Fig. 1 one uses the identity twice to delete the two interactions involving that site. Then one has

$$\langle\sigma_0\rangle = \frac{\langle\sigma_0\rangle_{01,\,02} + T\langle\sigma_2\rangle_{01,\,02}}{1 + T\langle\sigma_0\sigma_2\rangle_{01,\,02}} + \frac{T\langle\sigma_1\rangle_{01}}{1 + T\langle\sigma_0\sigma_1\rangle_{01}} \tag{4}$$

Now we are looking to find a condition on the temperature where we can prove that the left hand side of (4) is zero. Hence we will find an upper bound to the left hand side of (4) which we can eventually show goes to zero. All thermal averages are non-negative by the first Griffiths, Kelly, Sherman correlation inequality[4, 5, 6] (hereafter GKS I) and hence if we want an upper bound to the terms on the right hand side of (4) we can begin by setting each denominator equal to 1. Furthermore if we "un-delete" the interactions between pairs of site (0, 1) and (0, 2) by GKS II we further increase the right hand side therefore we have

$$\langle\sigma_0\rangle \leqslant 2T\langle\sigma_1\rangle \tag{5}$$

Note we have used the fact that $\langle\sigma_0\rangle_{01,\,02}$ and by symmetry $\langle\sigma_1\rangle = \langle\sigma_2\rangle$. Now we delete the five interactions involving σ_1. Note that there are actually 2 interactions between sites 1 and 2 due to our statement at the start of the proof regarding an extra interaction between the right most and left most sites on any horizontal line. Now we make the same approximations used in going from Eq. (4) to Eq. (5). Doing so gives us

$$\langle\sigma_0\rangle < 2T\langle\sigma_1\rangle < 2T(T(\langle\sigma_0\rangle + \langle\sigma_2\rangle + \langle\sigma_3\rangle + \langle\sigma_4\rangle)) \tag{6}$$

Location of sites 0, 1, 2, 3, and 4 are denoted in Fig. 1a. If we have that $\langle\sigma_0\rangle \leqslant \langle\sigma_1\rangle \leqslant \langle\sigma_4\rangle$ then we can write

$$\langle\sigma_0\rangle \leqslant 2T\langle\sigma_1\rangle \leqslant 2T(5T)\langle\sigma_4\rangle \tag{7}$$

We can continue this process ad infinitum where at each step we must replace spins at lower levels with a spin at higher level. As long as we know that the thermal average of a spin at the higher level is greater than or equal to that of a spin at a lower level we are always producing an upper bound on the right hand side. We do know this by GKS II since we can start with the root site spin, add interactions and have it become a spin at the first level. Since we have added interactions by GKS II as a spin at the first level it has a thermal average equal to or greater than its initial value which was as the root site spin. By adding still more interactions we can change our spin from a first level spin to a second level spin and again by GKS II it then must have a thermal average equal to or greater than what it had as a first level spin. Hence after N steps we have established

$$\langle \sigma_0 \rangle \leqslant 2T \langle \sigma_1 \rangle \leqslant 2T(5T)^N \langle \sigma_n \rangle \tag{8}$$

letting $N \to \infty$ we have, if $5T < 1$, then $\langle \sigma_0 \rangle \leqslant 0$. By GKS I $\langle \sigma_0 \rangle \geqslant 0$, hence $\langle \sigma_0 \rangle = 0$. For general k we have, if $(k+3)\, T < 1$, then $\langle \sigma_0 \rangle = 0$ and the proposition is proven.

See Table 1 for comparison of this bound with that of ref. 1. The inequality $\tanh(\beta J) < 1/(k+3)$ is equivalent to $\beta J < (1/2) \ln[(k+4)/(k+2)]$. Hence we have that a Bethe lattice with branching ratio k has a critical temperature which is a lower bound for these graphs and a Bethe lattice with branding ratio $k+3$ has a critical temperature which is an upper bound for these graphs.

REFERENCES

1. C. C. Wu, Ising Models on Hyperbolic Graphs, *J. Stat. Phys.* **50**:251 (1996).
2. T. P. Eggarter, Cayley trees, the Ising problem, and the thermodynamic limit, *Phys. Rev. B.* **9**:2989 (1974).
3. C. J. Thompson, Local properties of an Ising model on a Cayley tree, *J. Stat. Phys.* **27**:441 (1982).
4. R. B. Griffiths, Correlations in Ising ferromagnets I and II, *J. Math. Phys.* **8**:478, 484 (1967).
5. D. G. Kelly, and S. Sherman, General Griffiths Inequalities on correlations in Ising ferromagnets, *J. Math. Phys.* **9**:313 (1968).
6. G. S. Sylvester, Inequalities for continuous spin Ising ferromagnets, *J. Stat. Phys.* **15**:327 (1976).
7. J. L. Monroe, Upper bounds on the critical temperature for various Ising models, *J. Stat. Phys.* **40**:249 (1985).

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